

# Inverse Problems for Ultrahyperbolic Schrödinger Equations

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## ABSTRACT

**Abstract.** In this paper, we establish a global Carleman estimate for an Ultrahyperbolic Schrödinger equation. Moreover, we prove Hölder stability for the inverse problem of determining a coefficient or a source term in the Ultrahyperbolic Schrödinger equation by some lateral boundary data.

**Keywords.** Ultrahyperbolic Schrödinger equation, Inverse problem, Stability, Carleman estimate

## 1. INTRODUCTION

Let  $n, m \in \mathbb{N}$ ,  $T > 0$  and let  $D \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\partial D$  and  $G = \{y \in \mathbb{R}^m; |y| < L\}$  for  $L > 0$ .

We set  $Q = D \times G \times (0, T)$ ,  $\Sigma = \partial D \times G \times (0, T)$  and  $i = \sqrt{-1}$ .

We consider the ultrahyperbolic Schrödinger equation

$$i\partial_t v(x, y, t) + \Delta_y v(x, y, t) - \Delta_x v(x, y, t) - p(x, y)v(x, y, t) = 0, \quad (x, y, t) \in Q, \quad (1.1)$$

with the following initial and Dirichlet boundary data

$$v(x, y, 0) = a(x, y), \quad (x, y) \in D \times G, \quad (1.2)$$

$$v(x, y, t) = 0, \quad (x, y, t) \in \Sigma. \quad (1.3)$$

Throughout this paper, we use the following notations:

$$\begin{aligned}\partial_t &= \frac{\partial}{\partial t}, \quad \partial_{x_i} = \frac{\partial}{\partial x_i}, \quad \partial_{y_j} = \frac{\partial}{\partial y_j}, \quad \nabla_x = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}), \\ \nabla_y &= (\partial_{y_1}, \partial_{y_2}, \dots, \partial_{y_m}), \quad \Delta_x = \sum_{i=1}^n \partial_{x_i}^2, \quad \Delta_y = \sum_{j=1}^m \partial_{y_j}^2.\end{aligned}$$

Let  $v = v(p)$  satisfy (1.1)–(1.3). We discuss the following coefficient inverse problem.

**Problem 1** Determine the coefficient  $p(x, y)$ ,  $(x, y) \in D \times G$  in (1.1)–(1.3) by the extra data  $\partial_\nu v(p)|_\Sigma$ , where  $\nu \in \mathbb{R}^n$  denotes the unit outward normal vector to  $\partial D$  and  $\partial_\nu v = \nabla v \cdot \nu$  is the normal derivative.

Ultrahyperbolic Schrödinger equations arise in several applications, for example in water wave problems, [10, 11, 14, 34, 35] and in higher dimensions as completely integrable models, see [1, 25]. There have been limited number of studies on the direct problems for these equations. The local well posedness of the initial value problem was investigated in [19–21]. To our best knowledge there is no result available in the mathematical literature related to the inverse problems for ultrahyperbolic Schrödinger equations. In this work, we obtain a Carleman estimate and prove conditional Hölder stability for the inverse problem of determining a coefficient or a source term in ultrahyperbolic Schrödinger equation.

T. Carleman [8] established the first Carleman estimate in 1939 for proving the unique continuation for a two-dimensional elliptic equation. In 1954, C. Müller extended Carleman’s result to  $\mathbb{R}^n$ , [18]. After that A. P. Calderón [7] and L. Hörmander [13] improved these results based on the concept of pseudo-convexity.

In the theory of inverse problems, Carleman estimates were firstly introduced by A. L. Bukgeim and M. V. Klibanov in [6]. After that, there have been many works relying on that method with modified arguments. Puel and Yamamoto [28], Isakov and Yamamoto [17], Imanuvilov and Yamamoto [15, 16], Bellassoued and Yamamoto [5], Klibanov and Yamamoto [24] have obtained various stability estimates for inverse problems for hyperbolic equation.

We refer to Yamamoto [31] for a comprehensive survey about the stability and

observability results for inverse problems for parabolic equations.

Inverse problems for ultrahyperbolic equations were considered in [2, 26, 29], where the unique continuation and stability were proved by using the Carleman estimates. Gölgeleyen and Yamamoto [12] proved conditional Hölder stability for some inverse problems for ultrahyperbolic equation.

If  $n = 0$  then (1.1) is a classical Schrödinger equation which describes the evaluation of wave function of a charged particle under the influence of electrical potential  $p$ . As for the classical Schrödinger equation, Baudouin and Puel in [3] established a global Carleman estimate and proved the uniqueness and Lipschitz stability based on the idea by Imanuvilov and Yamamoto [16]. This result was improved by Mercado et al. [27] under a relaxed pseudoconvexity condition. In [3, 27], the main assumption is that the part of the boundary where the measurement is made satisfies a geometric condition related to geometric optics condition for the observability. This geometric condition was relaxed in Bellassoued and Choulli [4] under the assumption that the potential is known in a neighborhood of the boundary of the spatial domain. Yuan and Yamamoto [33] obtained a Carleman estimate with a regular weight function in Sobolev spaces of negative orders. They proved the uniqueness in the inverse problem of determining  $L^p$  potentials and obtained an  $L^2$  level observability inequality and unique continuation results for the Schrödinger equation. Cristofol and Soccorsi [9] considered the inverse problem of determining time-dependent coefficient of the Schrödinger equation from a finite number of Neumann data. Kian et al. [22] extended the stability result of [3] to the case of unbounded domains.

This paper consists of four sections. The rest of the paper is organized as follows. In Section 2, the main result of this paper (Theorem 1) is presented. In Section 3, a Carleman estimate (Proposition 1) which will be used in the proof of our main result is established. Finally, Section 4 is devoted to the proof of Theorem 1.

## 2. MAIN RESULT

Problem 1 can be reduced to an inverse source problem. For this aim, let  $v(p)$  and  $v(q)$  be two solutions of (1.1)–(1.3) with the coefficients  $p$  and  $q$  respectively.

Then the difference  $u = v(p) - v(q)$  satisfies

$$\begin{aligned} Au &= i\partial_t u(x, y, t) + \Delta_y u(x, y, t) - \Delta_x u(x, y, t) - p(x, y)u(x, y, t) \\ &= f(x, y)R(x, y, t), \quad (x, y, t) \in Q, \end{aligned} \quad (2.1)$$

$$u(x, y, 0) = 0, \quad (x, y) \in D \times G, \quad (2.2)$$

$$u(x, y, t) = 0, \quad (x, y, t) \in \Sigma \quad (2.3)$$

with  $f(x, y) = p(x, y) - q(x, y)$  and  $R = v(q)(x, y, t)$ .

We consider the following inverse source problem:

**Problem 2** Let  $p, R$  be given suitably. Then determine  $f(x, y)$ ,  $(x, y) \in D \times G$  by the extra data  $\partial_\nu u|_\Sigma$ .

Here we do not assume the uniqueness of  $v(p)$  and  $v(q)$  but their existence. We have the following result for Problem 2.

**Theorem 1** Let  $p \in L^\infty(D \times \{|y| < 2L\})$  and  $u$  satisfy (2.1)–(2.3) in  $D \times \{|y| < 2L\} \times (-T, T)$ .

We assume that

$$u = 0 \text{ on } \partial D \times \{|y| < 2L\} \times (-T, T),$$

$$\|\partial_t^k u\|_{H^2(D \times \{|y| < 2L\} \times (-T, T))} \leq M, \quad k = 1, 2 \text{ and}$$

$$R(x, y, 0) \in \mathbb{R} \text{ or } iR(x, y, 0) \in \mathbb{R} \text{ a. e. in } (D \times \{|y| < 2L\}),$$

$$R \in H^2(-T, T; L^\infty(D \times \{|y| < 2L\})),$$

$$\|\partial_t^k R\|_{L^2(-T, T; L^\infty(D \times \{|y| < 2L\}))} \leq M, \quad k = 1, 2.$$

We further assume that there exists a constant  $r_0 > 0$  such that

$$|R(x, y, 0)| \geq r_0, \quad x \in \overline{D}, \quad |y| \leq 2L$$

and  $\alpha > 0$  is sufficiently small and

$$L > \frac{1}{\sqrt{\alpha}} \max_{x \in \overline{D}} |x - x_0|.$$

Then for any small  $\epsilon > 0$ , for all real valued  $f \in L^2(D \times \{|y| < L\})$  there exist constants  $C > 0$  and  $\theta \in (0, 1)$  depending on  $\epsilon, M, x_0$ , such that

$$\|f\|_{L^2(D \times \{|y| < L - \epsilon\})} \leq C \sum_{k=1}^2 \|\partial_\nu \partial_t^k u\|_{L^2(\partial D_+ \times \{|y| < 2L\} \times (-T, T))}^\theta.$$

### 3. KEY CARLEMAN ESTIMATE

In this section, we show a Carleman estimate for the ultrahyperbolic Schrödinger equation which will be used in the proof of our main result.

Let  $\Omega = D \times G \times (-T, T)$ ,  $\Gamma_x = \partial D \times G \times (-T, T)$ ,  $\Gamma_y = D \times \partial G \times (-T, T)$  and let  $\partial\Omega = \Gamma_x \cup \Gamma_y$ .

Let  $x_0 \notin \overline{D}$ ,  $y_0 \in \mathbb{R}^m$  and  $\alpha, \beta \in (0, 1)$ , we set the weight function

$$\varphi(x, y, t) = e^{\gamma \psi(x, y, t)}, \tag{3.1}$$

where

$$\psi(x, y, t) = |x - x_0|^2 - \alpha |y - y_0|^2 - \beta |t|^2, \tag{3.2}$$

$\gamma > 0$  is a parameter. Moreover, we set the geometric condition

$$\partial D_+ = \{x \in \partial D; (x - x_0) \cdot \nu \geq 0\} \tag{3.3}$$

for  $x_0 \notin \overline{D}$ .

We set

$$\begin{aligned} Lu := & i\partial_t u(x, y, t) + \Delta_y u(x, y, t) - \Delta_x u(x, y, t) + \sum_{i=1}^n a_i(x, y, t) u_{x_i} \\ & + \sum_{j=1}^m b_j(x, y, t) u_{y_j} + a_0(x, y, t) u, \end{aligned} \quad (3.4)$$

where  $a_i, b_j \in L^\infty(\Omega)$ ,  $0 \leq i \leq n$ ,  $1 \leq j \leq m$ .

**Proposition 1** Let us assume that  $0 < \alpha, \beta < 1$  be small and  $\gamma > 0$  be sufficiently large, and let

$$|x - x_0|^2 - \alpha^2 |y|^2 - \beta^2 |t|^2 > \delta_0^2, \quad (x, y, t) \in \Omega \quad (3.5)$$

with some  $\delta_0 > 0$ . Then there exist constants  $C > 0$  and  $s_0 > 0$  such that

$$\begin{aligned} & \int_{\Omega} (s |\nabla_y u|^2 + s |\nabla_x u|^2 + s^3 |u|^2) e^{2s\varphi} dx dy dt \\ \leq & C \int_{\Omega} |Lu|^2 e^{2s\varphi} dx dy dt + C \int_{\partial D_+ \times G \times (-T, T)} s |\partial_\nu u|^2 e^{2s\varphi} dS_x dy dt \end{aligned}$$

for all  $s \geq s_0$ , provided that

$$\begin{aligned} Lu & \in L^2(\Omega), \quad u \in H^2(\Omega), \\ u(x, y, t) &= 0, \quad (x, y, t) \in \Gamma_x, \\ u(x, y, t) &= |\nabla_y u(x, y, t)| = 0, \quad (x, y, t) \in \Gamma_y, \\ u(x, y, T) &= u(x, y, -T) = 0, \quad (x, y) \in D \times G. \end{aligned} \quad (3.6)$$

**Proof** Let us set

$$L_0 u := i\partial_t u + \Delta_y u - \Delta_x u = F, \quad (3.7)$$

and

$$z(x, y, t) = e^{s\varphi} u(x, y, t), \quad P_s z(x, y, t) = e^{s\varphi} L_0 u. \quad (3.8)$$

By (3.8), we calculate

$$\begin{aligned}
P_s z &= e^{s\varphi} L_0 u \\
&= i\partial_t z - is\partial_t \varphi z + \Delta_y z - \Delta_x z - 2s(\nabla_y \varphi \cdot \nabla_y z - \nabla_x \varphi \cdot \nabla_x z) \\
&\quad - s(\Delta_y \varphi - \Delta_x \varphi) z + s^2 \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) z.
\end{aligned}$$

Then we have

$$P_s z + is\partial_t \varphi z = P_s^+ z + P_s^- z, \quad (3.9)$$

where

$$P_s^+ z = i\partial_t z + \Delta_y z - \Delta_x z + s^2 \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) z, \quad (3.10)$$

$$P_s^- z = -2s(\nabla_y \varphi \cdot \nabla_y z - \nabla_x \varphi \cdot \nabla_x z) - s(\Delta_y \varphi - \Delta_x \varphi) z, \quad (3.11)$$

with the conventions  $z \cdot z' = \sum_{i=1}^N z_i z'_i$  for all  $z = (z_1, \dots, z_N) \in \mathbb{C}^N$ ,  $z' = (z'_1, \dots, z'_N) \in \mathbb{C}^N$ .

Then we have

$$\|P_s z + is\partial_t \varphi z\|_{L^2(\Omega)}^2 = \|P_s^+ z\|_{L^2(\Omega)}^2 + \|P_s^- z\|_{L^2(\Omega)}^2 + 2 \operatorname{Re} (P_s^+ z, P_s^- z)_{L^2(\Omega)}, \quad (3.12)$$

where  $\operatorname{Re}(z)$  is the real part of  $z$ . Now, we calculate the last term in (3.12) by using (3.10) and (3.11), we obtain

$$2 \operatorname{Re} (P_s^+ z, P_s^- z)_{L^2(\Omega)} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8, \quad (3.13)$$

where

$$\begin{aligned}
I_1 &= -4s \operatorname{Re} \int_{\Omega} i \partial_t z (\nabla_y \varphi \cdot \nabla_y \bar{z} - \nabla_x \varphi \cdot \nabla_x \bar{z}) dx dy dt, \\
I_2 &= -2s \operatorname{Re} \int_{\Omega} i \partial_t z (\Delta_y \varphi - \Delta_x \varphi) \bar{z} dx dy dt, \\
I_3 &= -4s \operatorname{Re} \int_{\Omega} \Delta_y z (\nabla_y \varphi \cdot \nabla_y \bar{z} - \nabla_x \varphi \cdot \nabla_x \bar{z}) dx dy dt, \\
I_4 &= -2s \operatorname{Re} \int_{\Omega} \Delta_y z (\Delta_y \varphi - \Delta_x \varphi) \bar{z} dx dy dt, \\
I_5 &= 4s \operatorname{Re} \int_{\Omega} \Delta_x z (\nabla_y \varphi \cdot \nabla_y \bar{z} - \nabla_x \varphi \cdot \nabla_x \bar{z}) dx dy dt, \\
I_6 &= 2s \operatorname{Re} \int_{\Omega} \Delta_x z (\Delta_y \varphi - \Delta_x \varphi) \bar{z} dx dy dt, \\
I_7 &= -4s^3 \operatorname{Re} \int_{\Omega} \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) z (\nabla_y \varphi \cdot \nabla_y \bar{z} - \nabla_x \varphi \cdot \nabla_x \bar{z}) dx dy dt, \\
I_8 &= -2s^3 \operatorname{Re} \int_{\Omega} \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) z (\Delta_y \varphi - \Delta_x \varphi) \bar{z} dx dy dt,
\end{aligned}$$

and  $\bar{z}$  is the conjugate of  $z$ .

Now, we shall estimate the terms  $I_k$ ,  $1 \leq k \leq 8$ , using the integration by parts and the condition  $z(x, y, \pm T) = 0$ . Then we have

$$\begin{aligned}
I_1 &= -4s \operatorname{Re} \int_{\Omega} i \partial_t z \nabla_y \varphi \cdot \nabla_y \bar{z} dx dy dt + 4s \operatorname{Re} \int_{\Omega} i \partial_t z \nabla_x \varphi \cdot \nabla_x \bar{z} dx dy dt \\
&= -2s \operatorname{Im} \int_{\Omega} z \partial_t (\nabla_y \varphi) \cdot \nabla_y \bar{z} dx dy dt - 2s \operatorname{Im} \int_{\Gamma_y} z \partial_t \bar{z} (\nabla_y \varphi \cdot \nu) dS_y dx dt \\
&\quad + 2s \operatorname{Im} \int_{\Omega} z \Delta_y \varphi \partial_t \bar{z} dx dy dt + 2s \operatorname{Im} \int_{\Omega} z \partial_t (\nabla_x \varphi) \cdot \nabla_x \bar{z} dx dy dt \\
&\quad + 2s \operatorname{Im} \int_{\Gamma_x} z \partial_t \bar{z} (\nabla_x \varphi \cdot \nu) dS_x dy dt - 2s \operatorname{Im} \int_{\Omega} z \Delta_x \varphi \partial_t \bar{z} dx dy dt. \quad (3.14)
\end{aligned}$$

In (3.14), we used the equality  $\operatorname{Re}(iz) = -\operatorname{Im}(z)$  and  $\operatorname{Im}(z) - \operatorname{Im}(\bar{z}) = 2\operatorname{Im}(z)$ , where  $\operatorname{Im}(z)$  denotes the imaginary part of  $z \in \mathbb{C}$ .

$$\begin{aligned}
I_2 &= -2s \operatorname{Re} \int_{\Omega} i \partial_t z \bar{z} (\Delta_y \varphi - \Delta_x \varphi) dx dy dt \\
&= -2s \operatorname{Im} \int_{\Omega} \partial_t \bar{z} z (\Delta_y \varphi - \Delta_x \varphi) dx dy dt. \quad (3.15)
\end{aligned}$$



In (3.15), we used the equality  $\operatorname{Re}(iz) = \operatorname{Im}(\bar{z})$ .

$$\begin{aligned}
I_3 &= -4s \operatorname{Re} \int_{\Omega} \Delta_y z \nabla_y \varphi \cdot \nabla_y \bar{z} dx dy dt + 4s \operatorname{Re} \int_{\Omega} \Delta_y z \nabla_x \varphi \cdot \nabla_x \bar{z} dx dy dt \\
&= 4s \operatorname{Re} \int_{\Omega} \sum_{i,j=1}^n \varphi_{y_i y_j} \bar{z}_{y_i} z_{y_j} dx dy dt - 2s \int_{\Omega} \Delta_y \varphi |\nabla_y z|^2 dx dy dt \\
&\quad + 2s \int_{\Gamma_y} (\partial_{\nu} \varphi) |\nabla_y z|^2 dS_y dx dt - 4s \operatorname{Re} \int_{\Gamma_y} (\partial_{\nu} z) \nabla_y \varphi \cdot \nabla_y \bar{z} dS_y dx dt \\
&\quad - 4s \operatorname{Re} \int_{\Omega} \sum_{j=1}^m \sum_{i=1}^n \varphi_{x_j y_i} \bar{z}_{x_j} z_{y_i} dx dy dt + 2s \int_{\Omega} \Delta_x \varphi |\nabla_y z|^2 dx dy dt \\
&\quad - 2s \int_{\Gamma_x} (\partial_{\nu} \varphi) |\nabla_y z|^2 dS_x dy dt \\
&\quad + 4s \operatorname{Re} \int_{\Gamma_y} (\partial_{\nu} z) \nabla_x \varphi \cdot \nabla_x \bar{z} dS_y dx dt.
\end{aligned} \tag{3.16}$$

In (3.16), we used the equality  $\operatorname{Re} z_{y_j} \bar{z}_{y_i y_j} = \frac{1}{2} \left( |z_{y_j}|^2 \right)_{y_i}$ .

$$\begin{aligned}
I_4 &= -2s \operatorname{Re} \int_{\Omega} \Delta_y z (\Delta_y \varphi - \Delta_x \varphi) \bar{z} dx dy dt \\
&= -s \operatorname{Re} \int_{\Omega} \Delta_y (\Delta_y \varphi - \Delta_x \varphi) |z|^2 dx dy dt \\
&\quad + 2s \int_{\Omega} (\Delta_y \varphi - \Delta_x \varphi) |\nabla_y z|^2 dx dy dt \\
&\quad + s \int_{\Gamma_y} \partial_{\nu} (\Delta_y \varphi - \Delta_x \varphi) |z|^2 dS_y dx dt \\
&\quad - 2s \operatorname{Re} \int_{\Gamma_y} (\partial_{\nu} z) (\Delta_y \varphi - \Delta_x \varphi) \bar{z} dS_y dx dt.
\end{aligned} \tag{3.17}$$

In (3.17), we used the equality  $\operatorname{Re} \bar{z} \nabla_y z = \frac{1}{2} \nabla_y (|z|^2)$ .

$$\begin{aligned}
I_5 &= 4s \operatorname{Re} \int_{\Omega} \Delta_x z \nabla_y \varphi \cdot \nabla_y \bar{z} dx dy dt - 4s \operatorname{Re} \int_{\Omega} \Delta_x z \nabla_x \varphi \cdot \nabla_x \bar{z} dx dy dt \\
&= -4s \operatorname{Re} \int_{\Omega} \sum_{j=1}^m \sum_{i=1}^n \varphi_{y_i x_j} \bar{z}_{y_i} z_{x_j} dx dy dt - 2s \int_{\Omega} \sum_{i=1}^m \Delta_y \varphi |\nabla_x z|^2 dx dy dt \\
&\quad - 2s \int_{\Gamma_y} (\partial_{\nu} \varphi) |\nabla_x z|^2 dS_y dx dt + 4s \operatorname{Re} \int_{\Gamma_x} (\partial_{\nu} z) \nabla_y \varphi \cdot \nabla_y \bar{z} dS_x dy dt \\
&\quad + 4s \operatorname{Re} \int_{\Omega} \sum_{i,j=1}^n \varphi_{x_i y_j} \bar{z}_{x_i} z_{y_j} dx dy dt - 2s \int_{\Omega} \Delta_x \varphi |\nabla_x z|^2 dx dy dt \\
&\quad - 2s \int_{\Gamma_x} (\partial_{\nu} \varphi) |\nabla_x z|^2 dS_x dy dt \\
&\quad + 4s \operatorname{Re} \int_{\Gamma_x} (\partial_{\nu} z) \nabla_x \varphi \cdot \nabla_x \bar{z} dS_x dy dt.
\end{aligned} \tag{3.18}$$

In (3.18), we used the equality  $\operatorname{Re} z_{x_j} \bar{z}_{x_j y_i} = \frac{1}{2} (|z_{x_j}|^2)_{y_i}$ .

$$\begin{aligned}
I_6 &= 2s \operatorname{Re} \int_{\Omega} \Delta_x z (\Delta_y \varphi - \Delta_x \varphi) \bar{z} dx dy dt \\
&= -2s \int_{\Omega} (\Delta_y \varphi - \Delta_x \varphi) |\nabla_x z|^2 dx dy dt \\
&\quad + s \int_{\Omega} \Delta_x (\Delta_y \varphi - \Delta_x \varphi) |z|^2 dx dy dt \\
&\quad - s \int_{\Gamma_x} \partial_{\nu} (\Delta_y \varphi - \Delta_x \varphi) |z|^2 dS_x dy dt \\
&\quad + 2s \operatorname{Re} \int_{\Gamma_x} (\partial_{\nu} z) (\Delta_y \varphi - \Delta_x \varphi) \bar{z} dS_x dy dt.
\end{aligned} \tag{3.19}$$

In (3.19), we used the equality  $\operatorname{Re} \bar{z} \nabla_x z = \frac{1}{2} \nabla_x (|z|^2)$ .

$$\begin{aligned}
I_7 &= -4s^3 \operatorname{Re} \int_{\Omega} \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) z \left( \nabla_y \varphi \cdot \nabla_y \bar{z} - \nabla_x \varphi \cdot \nabla_x \bar{z} \right) dx dy dt \\
&= 2s^3 \int_{\Omega} |z|^2 \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) (\Delta_y \varphi - \Delta_x \varphi) dx dy dt \\
&\quad - 2s^3 \int_{\Gamma_y} (\partial_\nu \varphi) |z|^2 \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) dS_y dx dt \\
&\quad + 2s^3 \int_{\Omega} |z|^2 \nabla_y \varphi \cdot \nabla_y \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) dx dy dt \\
&\quad + 2s^3 \int_{\Gamma_x} \partial_\nu \varphi |z|^2 \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) dS_x dy dt \\
&\quad - 2s^3 \int_{\Omega} |z|^2 \nabla_x \varphi \cdot \nabla_x \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) dx dy dt. \tag{3.20}
\end{aligned}$$

In (3.20), we used the equality  $\operatorname{Re} z \nabla_y \bar{z} = \frac{1}{2} (\nabla_y |z|^2)$  and  $\operatorname{Re} z \nabla_x \bar{z} = \frac{1}{2} (\nabla_x |z|^2)$ .

$$\begin{aligned}
I_8 &= -2s^3 \operatorname{Re} \int_{\Omega} \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) (\Delta_y \varphi - \Delta_x \varphi) z \bar{z} dx dy dt \\
&= -2s^3 \int_{\Omega} \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) (\Delta_y \varphi - \Delta_x \varphi) |z|^2 dx dy dt. \tag{3.21}
\end{aligned}$$

Hence, we can rewrite (3.13)

$$2 \operatorname{Re} (P_s^+ z, P_s^- z)_{L^2(\Omega)} = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + B_0,$$

where

$$\begin{aligned}
J_1 &= -2s \operatorname{Im} \int_{\Omega} z \partial_t (\nabla_y \varphi) \cdot \nabla_y \bar{z} dx dy dt + 2s \operatorname{Im} \int_{\Omega} z \partial_t (\nabla_x \varphi) \cdot \nabla_x \bar{z} dx dy dt, \\
J_2 &= 4s \operatorname{Re} \sum_{i,j=1}^n \int_{\Omega} \varphi_{y_i y_j} z_{y_j} \bar{z}_{y_i} dx dy dt - 4s \operatorname{Re} \sum_{j=1}^m \sum_{i=1}^n \int_{\Omega} \varphi_{y_i x_j} z_{y_i} \bar{z}_{x_j} dx dy dt, \\
J_3 &= -s \int_{\Omega} |z|^2 \Delta_y (\Delta_y \varphi - \Delta_x \varphi) dx dy dt, \\
J_4 &= -4s \operatorname{Re} \sum_{j=1}^m \sum_{i=1}^n \int_{\Omega} \varphi_{y_i x_j} \bar{z}_{y_i} z_{x_j} dx dy dt + 4s \operatorname{Re} \sum_{i,j=1}^n \int_{\Omega} \varphi_{x_i x_j} \bar{z}_{x_i} z_{x_j} dx dy dt,
\end{aligned}$$

$$\begin{aligned}
J_5 &= s \int_{\Omega} |z|^2 \Delta_x (\Delta_y \varphi - \Delta_x \varphi) dx dy dt, \\
J_6 &= 2s^3 \int_{\Omega} |z|^2 \nabla_y \varphi \cdot \nabla_y \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) dx dy dt \\
&\quad - 2s^3 \int_{\Omega} |z|^2 \nabla_x \varphi \cdot \nabla_x \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) dx dy dt
\end{aligned}$$

and

$$\begin{aligned}
B_0 &= -2s \operatorname{Im} \int_{\Gamma_y} z \partial_t \bar{z} (\nabla_y \varphi \cdot \nu) dS_y dx dt \\
&\quad + 4s \operatorname{Re} \int_{\Gamma_y} (\partial_\nu z) (\nabla_x \varphi \cdot \nabla_x \bar{z} - \nabla_y \varphi \cdot \nabla_y \bar{z}) dS_y dx dt \\
&\quad + 2s \int_{\Gamma_y} (\partial_\nu \varphi) \left( |\nabla_y z|^2 - |\nabla_x z|^2 \right) dS_y dx dt \\
&\quad - 2s^3 \int_{\Gamma_y} (\partial_\nu \varphi) |z|^2 \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) dS_y dx dt \\
&\quad - 2s \int_{\Gamma_y} (\partial_\nu z) (\Delta_y \varphi - \Delta_x \varphi) \bar{z} dS_y dx dt + s \int_{\Gamma_y} \partial_\nu (\Delta_y \varphi - \Delta_x \varphi) |z|^2 dS_y dx dt \\
&\quad + 2s \operatorname{Im} \int_{\Gamma_x} z \partial_t \bar{z} (\nabla_x \varphi \cdot \nu) dS_x dy dt \\
&\quad + 4s \operatorname{Re} \int_{\Gamma_x} (\partial_\nu z) (\nabla_y \varphi \cdot \nabla_y \bar{z} - \nabla_x \varphi \cdot \nabla_x \bar{z}) dS_x dy dt \\
&\quad - 2s \int_{\Gamma_x} (\partial_\nu \varphi) \left( |\nabla_y z|^2 - |\nabla_x z|^2 \right) dS_x dy dt \\
&\quad + 2s^3 \int_{\Gamma_x} (\partial_\nu \varphi) |z|^2 \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) dS_x dy dt \\
&\quad + 2s \int_{\Gamma_x} (\partial_\nu z) (\Delta_y \varphi - \Delta_x \varphi) \bar{z} dS_x dy dt - s \int_{\Gamma_x} \partial_\nu (\Delta_y \varphi - \Delta_x \varphi) |z|^2 dS_x dy dt.
\end{aligned}$$

Next, we shall estimate  $J_k$ ,  $1 \leq k \leq 6$  and  $B_0$  using the following elementary properties of the weight function:

$$\begin{aligned}
\partial_t \varphi &= (-2\gamma\beta t) \varphi, & \varphi_{x_i x_i} &= \gamma \varphi (2 + \gamma \psi_{x_i}^2), \\
\varphi_{x_i y_j} &= \gamma^2 \varphi \psi_{x_i} \psi_{y_j}, & \varphi_{x_i x_j} &= \gamma \varphi \left( \psi_{x_i x_j} + \gamma \psi_{x_i} \psi_{x_j} \right), \\
\varphi_{y_i y_j} &= \gamma \varphi \left( \psi_{y_i y_j} + \gamma \psi_{y_i} \psi_{y_j} \right), & \nabla_x \varphi &= \gamma \varphi \nabla_x \psi, \\
\nabla_y \varphi &= \gamma \varphi \nabla_y \psi, & \partial_t (\nabla_x \varphi) &= (-2\gamma^2 \beta t) \varphi \nabla_x \psi, \\
\partial_t (\nabla_y \varphi) &= (-2\gamma^2 \beta t) \varphi \nabla_y \psi, & \Delta_x \varphi &= \gamma \varphi \left( \Delta_x \psi + \gamma |\nabla_x \psi|^2 \right), \\
\Delta_y \varphi &= \gamma \varphi \left( \Delta_y \psi + \gamma |\nabla_y \psi|^2 \right), & \Delta_y \varphi - \Delta_x \varphi &= \gamma \varphi d_1(\psi) + \gamma^2 \varphi d_2(\psi),
\end{aligned}$$

where

$$\begin{aligned} d_1(\psi) &= \Delta_y \psi - \Delta_x \psi, \\ d_2(\psi) &= |\nabla_y \psi|^2 - |\nabla_x \psi|^2. \end{aligned}$$

Then, we obtain

$$\begin{aligned} J_1 &= -2s \operatorname{Im} \int_{\Omega} z \partial_t (\nabla_y \varphi) \cdot \nabla_y \bar{z} dx dy dt + 2s \operatorname{Im} \int_{\Omega} z \partial_t (\nabla_x \varphi) \cdot \nabla_x \bar{z} dx dy dt \\ &= -2s \operatorname{Im} \int_{\Omega} (-2\gamma^2 \beta t) z \varphi \nabla_y \psi \cdot \nabla_y \bar{z} dx dy dt \\ &\quad + 2s \operatorname{Im} \int_{\Omega} (-2\gamma^2 \beta t) z \varphi \nabla_x \psi \cdot \nabla_x \bar{z} dx dy dt \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} J_2 &= \sum_{i,j=1}^m \operatorname{Re} \int_{\Omega} 4s \varphi_{y_i y_j} z_{y_j} \bar{z}_{y_i} dx dy dt - \operatorname{Re} \sum_{j=1}^m \sum_{i=1}^n \int_{\Omega} 4s z_{y_i} \bar{z}_{x_j} \varphi_{y_i x_j} dx dy dt \\ &= \sum_{i,j=1}^m \operatorname{Re} \int_{\Omega} 4s \gamma \varphi \left( \psi_{y_i y_j} + \gamma \psi_{y_i} \psi_{y_j} \right) z_{y_j} \bar{z}_{y_i} dx dy dt + \int_{\Omega} 4s \gamma^2 \varphi |\nabla_y \psi \cdot \nabla_y z|^2 dx dy dt \\ &\quad - \sum_{j=1}^m \sum_{i=1}^n \int_{\Omega} 4s \gamma^2 \varphi (\nabla_y \psi \cdot \nabla_y z) (\nabla_x \psi \cdot \nabla_x \bar{z}) dx dy dt. \end{aligned} \quad (3.23)$$

Before estimating  $J_3$ , we can directly verify

$$\begin{aligned} \Delta_y (\varphi d_2(\psi)) &= (\Delta_y \varphi) d_2(\psi) + 2 \nabla_y \varphi \cdot \nabla_y (d_2(\psi)) + \varphi \Delta_y (d_2(\psi)) \\ &= \gamma \varphi (\Delta_y \psi) d_2(\psi) + \gamma^2 \varphi |\nabla_y \psi|^2 d_2 \psi + 2 \gamma \varphi \nabla_y \psi \cdot \nabla_y (d_2(\psi)) \\ &\quad + \varphi \Delta_y (d_2(\psi)), \\ \Delta_y (\varphi d_1(\psi)) &= (\Delta_y \varphi) d_1(\psi) + 2 \nabla_y \varphi \cdot \nabla_y (d_1(\psi)) + \varphi \Delta_y (d_1(\psi)) \\ &= \gamma \varphi (\Delta_y \psi) d_1(\psi) + \gamma^2 \varphi |\nabla_y \psi|^2 d_1 \psi + 2 \gamma \varphi \nabla_y \psi \cdot \nabla_y (d_1(\psi)) \\ &\quad + \varphi \Delta_y (d_1(\psi)). \end{aligned}$$

Then, we have

$$\begin{aligned}
J_3 &= -s \int_{\Omega} |z|^2 \Delta_y (\Delta_y \varphi - \Delta_x \varphi) dx dy dt \\
&= - \int_{\Omega} s \gamma^2 \varphi |z|^2 (d_1(\psi) (\Delta_y \psi) + \Delta_y (d_2(\psi))) dx dy dt \\
&\quad - \int_{\Omega} s \gamma^3 \varphi |z|^2 \left( d_1(\psi) |\nabla_y \psi|^2 + (\Delta_y \psi) d_2(\psi) + 2 \nabla_y \psi \cdot \nabla_y (d_2(\psi)) \right) dx dy dt \\
&\quad - \int_{\Omega} s \gamma^4 \varphi |z|^2 |\nabla_y \psi|^2 d_2(\psi) dx dy dt, \tag{3.24}
\end{aligned}$$

$$\begin{aligned}
J_4 &= - \sum_{j=1}^m \sum_{i=1}^n \operatorname{Re} \int_{\Omega} 4s \varphi_{y_i x_j} \bar{z}_{y_i} z_{x_j} dx dy dt + \sum_{i,j=1}^n \operatorname{Re} \int_{\Omega} 4s \varphi_{x_i x_j} \bar{z}_{x_i} z_{x_j} dx dy dt \\
&= - \operatorname{Re} \int_{\Omega} 4s \gamma^2 \varphi (\nabla_y \psi \cdot \nabla_y \bar{z}) (\nabla_x \psi \cdot \nabla_x z) dx dy dt \\
&\quad + \sum_{i,j=1}^n \operatorname{Re} \int_{\Omega} 4s \gamma \varphi \psi_{x_i x_j} \bar{z}_{x_i} z_{x_j} dx dy dt \\
&\quad + \int_{\Omega} 4s \gamma^2 \varphi |\nabla_x \psi \cdot \nabla_x z|^2 dx dy dt. \tag{3.25}
\end{aligned}$$

Since

$$\begin{aligned}
\Delta_x (\varphi d_2(\psi)) &= (\Delta_x \varphi) d_2(\psi) + 2 \nabla_x \varphi \cdot \nabla_x (d_2(\psi)) + \varphi \Delta_x (d_2(\psi)) \\
&= \gamma \varphi (\Delta_x \psi) d_2(\psi) + \gamma^2 \varphi |\nabla_x \psi|^2 d_2(\psi) + 2 \gamma \varphi \nabla_x \psi \cdot \nabla_x (d_2(\psi)) \\
&\quad + \varphi \Delta_x (d_2(\psi)),
\end{aligned}$$

we see that

$$\begin{aligned}
J_5 &= s \int_{\Omega} |z|^2 \Delta_x (\Delta_y \varphi - \Delta_x \varphi) dx dy dt \\
&= \int_{\Omega} s \gamma^2 \varphi |z|^2 (d_1(\psi) (\Delta_x \psi) + \Delta_x (d_2(\psi))) dx dy dt \\
&\quad + \int_{\Omega} s \gamma^3 \varphi |z|^2 \left( d_1(\psi) |\nabla_x \psi|^2 + (\Delta_x \psi) d_2(\psi) + 2 \nabla_x \psi \cdot \nabla_x (d_2(\psi)) \right) dx dy dt \\
&\quad + \int_{\Omega} s \gamma^4 \varphi |z|^2 |\nabla_x \psi|^2 d_2(\psi) dx dy dt. \tag{3.26}
\end{aligned}$$

Since

$$\nabla_y \varphi \cdot \nabla_y \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) = 2\gamma^4 \varphi^3 d_2(\psi) \nabla_y \psi \cdot \nabla_y \psi + \gamma^3 \varphi^3 \nabla_y \psi \cdot \nabla_y (d_2(\psi))$$

and

$$\nabla_x \varphi \cdot \nabla_x \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) = 2\gamma^4 \varphi^3 d_2(\psi) \nabla_x \psi \cdot \nabla_x \psi + \gamma^3 \varphi^3 \nabla_x \psi \cdot \nabla_x (d_2(\psi)),$$

we have

$$\begin{aligned} J_6 &= 2s^3 \int_{\Omega} |z|^2 \nabla_y \varphi \cdot \nabla_y \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) dx dy dt \\ &\quad - 2s^3 \int_{\Omega} |z|^2 \nabla_x \varphi \cdot \nabla_x \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) dx dy dt \\ &= 4 \int_{\Omega} s^3 \gamma^4 \varphi^3 |z|^2 (d_2(\psi))^2 dx dy dt \\ &\quad + \int_{\Omega} 2s^3 \gamma^3 \varphi^3 |z|^2 \nabla_y \psi \cdot \nabla_y (d_2(\psi)) dx dy dt \\ &\quad - \int_{\Omega} 2s^3 \gamma^3 \varphi^3 |z|^2 \nabla_x \psi \cdot \nabla_x (d_2(\psi)) dx dy dt. \end{aligned} \quad (3.27)$$

Finally, the boundary term is obtained as follows:

$$\begin{aligned} B_0 &= -2 \operatorname{Im} \int_{\Gamma_y} s \gamma \varphi z \partial_t \bar{z} (\nabla_y \psi \cdot \nu) dS_y dx dt \\ &\quad + 4 \operatorname{Re} \int_{\Gamma_y} s \gamma \varphi (\partial_\nu z) (\nabla_x \psi \cdot \nabla_x \bar{z} - \nabla_y \psi \cdot \nabla_y \bar{z}) dS_y dx dt \\ &\quad + \int_{\Gamma_y} 2s \gamma \varphi (\partial_\nu \psi) \left( |\nabla_y z|^2 - |\nabla_x z|^2 \right) dS_y dx dt \\ &\quad - \int_{\Gamma_y} 2s^3 \gamma^3 \varphi^3 \partial_\nu \psi |z|^2 d_2(\psi) dS_y dx dt \\ &\quad - \int_{\Gamma_y} 2s (\gamma \varphi d_1(\psi) + \gamma^2 \varphi d_2(\psi)) \bar{z} (\partial_\nu z) dS_y dx dt \\ &\quad + \int_{\Gamma_y} s ((\gamma^2 \varphi d_1(\psi) + \gamma^3 \varphi d_2(\psi)) (\partial_\nu \psi) \\ &\quad + \gamma^2 \varphi (\partial_\nu (d_2(\psi)))) |z|^2 dS_y dx dt \\ &\quad + 2 \operatorname{Im} \int_{\Gamma_x} s \gamma \varphi z \partial_t \bar{z} (\nabla_x \psi \cdot \nu) dS_x dy dt \\ &\quad + 4 \operatorname{Re} \int_{\Gamma_x} s \gamma \varphi (\partial_\nu z) (\nabla_y \psi \cdot \nabla_y \bar{z} - \nabla_x \psi \cdot \nabla_x \bar{z}) dS_x dy dt \end{aligned}$$

$$\begin{aligned}
& - \int_{\Gamma_x} 2s\gamma\varphi(\partial_\nu\psi) \left( |\nabla_y z|^2 - |\nabla_x z|^2 \right) dS_x dy dt \\
& + \int_{\Gamma_x} 2s^3\gamma^3\varphi^3(\partial_\nu\psi) |z|^2 d_2(\psi) dS_x dy dt \\
& + \int_{\Gamma_x} 2s \left( \gamma\varphi d_1(\psi) + \gamma^2\varphi d_2(\psi) \right) \bar{z}(\partial_\nu z) dS_x dy dt \\
& - \int_{\Gamma_x} s \left( (\gamma^2\varphi d_1(\psi) + \gamma^3\varphi d_2(\psi)) (\partial_\nu\psi) \right. \\
& \left. + \gamma^2\varphi\partial_\nu(d_2(\psi)) \right) |z|^2 dS_x dy dt.
\end{aligned} \tag{3.28}$$

Then from (3.22)-(3.28) we can write

$$\begin{aligned}
2 \operatorname{Re} (P_s^+ z, P_s^- z)_{L^2(\Omega)} &= \sum_{i,j=1}^m \operatorname{Re} \int_{\Omega} 4s\gamma\varphi\psi_{y_i y_j} z_{y_j} \bar{z}_{y_i} dx dy dt \\
&+ \sum_{i,j=1}^n \operatorname{Re} \int_{\Omega} 4s\gamma\varphi\psi_{x_i x_j} \bar{z}_{x_i} z_{x_j} dx dy dt \\
&+ \int_{\Omega} 4s\gamma^2\varphi |\nabla_y \psi \cdot \nabla_y z - \nabla_x \psi \cdot \nabla_x z|^2 dx dy dt \\
&+ \int_{\Omega} 4s^3\gamma^4\varphi^3 |z|^2 (d_2(\psi))^2 dx dy dt + B_0 + X_1 + X_2,
\end{aligned}$$

where

$$X_1 = \int_{\Omega} 2s^3\gamma^3\varphi^3 |z|^2 d_5(\psi) dx dy dt,$$

$$\begin{aligned}
X_2 &= -2 \operatorname{Im} \int_{\Omega} s (-2\gamma^2\beta t) z\varphi \nabla_y \psi \cdot \nabla_y \bar{z} dx dy dt \\
&+ 2 \operatorname{Im} \int_{\Omega} s (-2\gamma^2\beta t) z\varphi \nabla_x \psi \cdot \nabla_x \bar{z} dx dy dt \\
&- \int_{\Omega} s\gamma^4\varphi |z|^2 (d_2(\psi))^2 dx dy dt \\
&- \int_{\Omega} s\gamma^2\varphi |z|^2 d_3(\psi) dx dy dt \\
&- \int_{\Omega} s\gamma^3\varphi |z|^2 d_4(\psi) dx dy dt,
\end{aligned}$$



and

$$\begin{aligned}
d_3 & : = d_3(\psi) = (d_1(\psi))^2 + \Delta_y(d_2(\psi)) - \Delta_x(d_2(\psi)), \\
d_4 & : = d_4(\psi) = d_1(\psi)d_2(\psi) + \nabla_y\psi \cdot \nabla_y(d_2(\psi)) - \nabla_x\psi \cdot \nabla_x(d_2(\psi)), \\
d_5 & : = d_5(\psi) = \nabla_y\psi \cdot \nabla_y(d_2(\psi)) - \nabla_x\psi \cdot \nabla_x(d_2(\psi)).
\end{aligned}$$

Since

$$\int_{\Omega} 4s\gamma^2\varphi |\nabla_y\psi \cdot \nabla_y z - \nabla_x\psi \cdot \nabla_x z|^2 dx dy dt \geq 0$$

and for  $0 < \alpha, \beta < 1$ ,

$$|x - x_0|^2 - \alpha^2|y|^2 - \beta^2|t|^2 \geq \delta_0^2,$$

we have

$$d_2^2 = 16(|x - x_0|^2 - \alpha^2|y - y_0|^2)^2 \geq 16(|x - x_0|^2 - \alpha^2|y - y_0|^2 - \beta^2|t|^2)^2 \geq 16\delta_0^2.$$

Then, we see that

$$\begin{aligned}
2 \operatorname{Re} (P_s^+ z, P_s^- z)_{L^2(\Omega)} & \geq - \int_{\Omega} 8s\alpha\gamma\varphi |\nabla_y z|^2 dx dy dt + \int_{\Omega} 8s\gamma\varphi |\nabla_x z|^2 dx dy dt \\
& \quad + 64\delta_0^2 \int_{\Omega} s^3\gamma^4\varphi^3 |z|^2 dx dy dt + B_0 + X_1 + X_2. \quad (3.29)
\end{aligned}$$

Since the signs of the terms of  $|\nabla_x z|^2$  and  $|\nabla_y z|^2$  are different, we need to perform

another estimation:

$$\begin{aligned}
2 \operatorname{Re} (P_s^+ z + P_s^- z, \varphi z)_{L^2(\Omega)} &= 2 \operatorname{Re} \int_{\Omega} i \partial_t z \bar{z} \varphi dx dy dt + 2 \operatorname{Re} \int_{\Omega} \Delta_y z \bar{z} \varphi dx dy dt \\
&\quad - 2 \operatorname{Re} \int_{\Omega} \Delta_x z \bar{z} \varphi dx dy dt \\
&\quad + 2 \operatorname{Re} \int_{\Omega} s^2 (|\nabla_y \varphi|^2 - |\nabla_x \varphi|^2) \varphi z \bar{z} dx dy dt \\
&\quad - 4 \operatorname{Re} \int_{\Omega} s (\nabla_y \varphi \cdot \nabla_y z - \nabla_x \varphi \cdot \nabla_x z) \varphi \bar{z} dx dy dt \\
&\quad - 2 \operatorname{Re} \int_{\Omega} s (\Delta_y \varphi - \Delta_x \varphi) z \varphi \bar{z} dx dy dt \\
&= K_1 + K_2 + K_3 + K_4 + K_5 + K_6.
\end{aligned}$$

Now we calculate the terms  $K_j$ ,  $1 \leq j \leq 6$  as follows:

$$\begin{aligned}
K_1 &= 2 \operatorname{Re} \int_{\Omega} i \partial_t z \bar{z} \varphi dx dy dt \\
&= -2 \operatorname{Im} \int_{\Omega} \partial_t z \bar{z} \varphi dx dy dt,
\end{aligned}$$

$$\begin{aligned}
K_2 &= 2 \operatorname{Re} \int_{\Omega} \Delta_y z \varphi \bar{z} dx dy dt \\
&= \int_{\Omega} \gamma \varphi |z|^2 \Delta_y \psi dx dy dt + \int_{\Omega} \gamma^2 \varphi |z|^2 |\nabla_y \psi|^2 dx dy dt \\
&\quad - 2 \int_{\Gamma_y} \gamma \varphi (\partial_\nu \psi) |z|^2 dS_y dx dt - 2 \int_{\Omega} \varphi |\nabla_y z|^2 dx dy dt \\
&\quad + 2 \operatorname{Re} \int_{\Gamma_y} (\partial_\nu z) (\varphi \bar{z}) dS_y dx dt,
\end{aligned}$$

$$\begin{aligned}
K_3 &= -2 \operatorname{Re} \int_{\Omega} \Delta_x z \bar{z} \varphi dx dy dt \\
&= 2 \int_{\Omega} \varphi |\nabla_x z|^2 dx dy dt - \int_{\Omega} \gamma \varphi \Delta_x \psi |z|^2 dx dy dt \\
&\quad - \int_{\Omega} \gamma^2 \varphi |\nabla_x \psi|^2 |z|^2 dx dy dt + \int_{\Gamma_x} \gamma \varphi (\partial_\nu \psi) |z|^2 dS_x dy dt \\
&\quad - 2 \operatorname{Re} \int_{\Gamma_x} (\partial_\nu z) (\varphi \bar{z}) dS_x dy dt,
\end{aligned}$$

$$\begin{aligned}
K_4 &= 2 \operatorname{Re} \int_{\Omega} s^2 \left( |\nabla_y \varphi|^2 - |\nabla_x \varphi|^2 \right) \varphi z \bar{z} dx dy dt \\
&= 2 \int_{\Omega} s^2 \gamma^2 \varphi^3 \left( |\nabla_y \psi|^2 - |\nabla_x \psi|^2 \right) |z|^2 dx dy dt \\
&= 2 \int_{\Omega} s^2 \gamma^2 \varphi^3 d_2(\psi) |z|^2 dx dy dt,
\end{aligned}$$

$$\begin{aligned}
K_5 &= -4 \operatorname{Re} \int_{\Omega} s \varphi \bar{z} (\nabla_y \varphi \cdot \nabla_y z - \nabla_x \varphi \cdot \nabla_x z) \varphi dx dy dt \\
&= 2 \int_{\Omega} s \varphi^2 \gamma |z|^2 d_1(\psi) dx dy dt + 4 \int_{\Omega} s \gamma^2 \varphi^2 |z|^2 d_2(\psi) dx dy dt \\
&\quad - 2 \int_{\Gamma_y} s \gamma \varphi^2 (\partial_\nu \psi) |z|^2 dS_y dx dt + 2 \int_{\Gamma_x} s \gamma \varphi^2 (\partial_\nu \psi) |z|^2 dS_x dy dt,
\end{aligned}$$

$$\begin{aligned}
K_6 &= -2 \operatorname{Re} \int_{\Omega} s \varphi z \bar{z} (\Delta_y \varphi - \Delta_x \varphi) dx dy dt \\
&= -2 \int_{\Omega} s \gamma \varphi^2 (\Delta_y \psi - \Delta_x \psi) |z|^2 dx dy dt \\
&\quad - 2 \int_{\Omega} s \gamma^2 \varphi^2 |z|^2 \left( |\nabla_y \psi|^2 - |\nabla_x \psi|^2 \right) dx dy dt \\
&= -2 \int_{\Omega} s \gamma \varphi^2 d_1(\psi) |z|^2 dx dy dt - 2 \int_{\Omega} s \gamma^2 \varphi^2 |z|^2 d_2(\psi) dx dy dt.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
2 \operatorname{Re} \int_{\Omega} (P_s^+ z + P_s^- z) \varphi \bar{z} dx dy dt &= -2 \int_{\Omega} \varphi |\nabla_y z|^2 dx dy dt + 2 \int_{\Omega} \varphi |\nabla_x z|^2 dx dy dt \\
&\quad + B_1 + X_3 + X_4,
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
X_3 &= 2 \int_{\Omega} s^2 \gamma^2 \varphi^3 d_2(\psi) |z|^2 dx dy dt, \\
X_4 &= -2 \operatorname{Im} \int_{\Omega} \partial_t z \bar{z} \varphi dx dy dt + \int_{\Omega} \gamma \varphi |z|^2 d_1(\psi) dx dy dt \\
&\quad + \int_{\Omega} s \gamma^2 \varphi |z|^2 d_2(\psi) dx dy dt + 2 \int_{\Omega} s \gamma^2 \varphi^2 |z|^2 d_2(\psi) dx dy dt.
\end{aligned}$$

We note that

$$\begin{aligned}
B_1 &= - \int_{\Gamma_y} \gamma \varphi (\partial_\nu \psi) |z|^2 dS_y dx dt + 2 \operatorname{Re} \int_{\Gamma_y} (\partial_\nu z) (\varphi \bar{z}) dS_y dx dt \\
&\quad - 2 \int_{\Gamma_y} s \gamma \varphi^2 (\partial_\nu \psi) |z|^2 dS_y dx dt + \int_{\Gamma_x} \gamma \varphi (\partial_\nu \psi) |z|^2 dS_x dy dt \\
&\quad - 2 \operatorname{Re} \int_{\Gamma_x} (\partial_\nu z) (\varphi \bar{z}) dS_x dy dt + 2 \int_{\Gamma_x} s \gamma \varphi^2 (\partial_\nu \psi) |z|^2 dS_x dx dt = 0,
\end{aligned}$$

since  $z = 0$  on  $\Gamma_x \cup \Gamma_y$ . We multiply (3.30) by  $-s\gamma(4\alpha + \mu)$ , then we have

$$\begin{aligned}
-2 \operatorname{Re} \int_{\Omega} (4\alpha + \mu) (P_s^+ z + P_s^- z) s \gamma \varphi \bar{z} dx dy dt &= 2 \int_{\Omega} (4\alpha + \mu) s \gamma \varphi |\nabla_y z|^2 dx dy dt \\
&\quad - 2 \int_{\Omega} (4\alpha + \mu) s \gamma \varphi |\nabla_x z|^2 dx dy dt \\
&\quad + X_5 + X_6, \tag{3.31}
\end{aligned}$$

where we choose  $\mu > 0$  later and

$$\begin{aligned}
X_5 &= -2 \int_{\Omega} (4\alpha + \mu) s^3 \gamma^3 \varphi^3 d_2(\psi) |z|^2 dx dy dt, \\
X_6 &= 2 \operatorname{Im} \int_{\Omega} s \gamma (4\alpha + \mu) \partial_t z \bar{z} \varphi dx dy dt \\
&\quad - \int_{\Omega} (4\alpha + \mu) s \gamma^2 \varphi |z|^2 d_1(\psi) dx dy dt \\
&\quad - \int_{\Omega} (4\alpha + \mu) s^2 \gamma^3 \varphi |z|^2 d_2(\psi) dx dy dt \\
&\quad - 2 \int_{\Omega} (4\alpha + \mu) s^2 \gamma^3 \varphi^2 |z|^2 d_2(\psi) dx dy dt.
\end{aligned}$$

By adding (3.29) and (3.31) we have

$$\begin{aligned}
&2 \operatorname{Re} (P_s^+ z, P_s^- z)_{L^2(\Omega)} - 2 \operatorname{Re} \int_{\Omega} (4\alpha + \mu) s \gamma (P_s^+ z + P_s^- z) \varphi \bar{z} dx dy dt \\
&\geq 2\mu \int_{\Omega} s \gamma \varphi |\nabla_y z|^2 dx dy dt + (8 - 8\alpha - 2\mu) \int_{\Omega} s \gamma \varphi |\nabla_x z|^2 dx dy dt \\
&\quad + 64\delta_0^2 \int_{\Omega} s^3 \gamma^4 \varphi^3 |z|^2 dx dy dt + B_0 + X_1 + X_2 + X_5 + X_6. \tag{3.32}
\end{aligned}$$

On the other hand, since

$$\|P_s z + is\partial_t \varphi z\|_{L^2(\Omega)}^2 = \|P_s^+ z\|_{L^2(\Omega)}^2 + \|P_s^- z\|_{L^2(\Omega)}^2 + 2 \operatorname{Re} (P_s^+ z, P_s^- z)_{L^2(\Omega)},$$

and

$$\begin{aligned} & -2 \operatorname{Re} \int_{\Omega} (4\alpha + \mu) s \gamma (P_s z + is\partial_t \varphi z) \varphi \bar{z} dx dy dt \\ & \leq (4\alpha + \mu) \int_{\Omega} |P_s z + is\partial_t \varphi z|^2 dx dy dt + C_1 s^2 \int_{\Omega} |z|^2 dx dy dt \end{aligned}$$

we have

$$\begin{aligned} C_2 \int_{\Omega} |P_s z + is\partial_t \varphi z|^2 dx dy dt & \geq \int_{\Omega} |P_s^+ z|^2 dx dy dt + \int_{\Omega} |P_s^- z|^2 dx dy dt \\ & \quad + 2s\gamma C_3 \int_{\Omega} \varphi |\nabla_x z|^2 dx dy dt + 2s\gamma C_4 \int_{\Omega} \varphi |\nabla_y z|^2 dx dy dt \\ & \quad + 64\delta_0^2 s^3 \gamma^4 \int_{\Omega} \varphi^3 |z|^2 dx dy dt + B_0 + X_1 + X_2 + X_5 + X_6. \end{aligned}$$

We see that there exists a constant  $\gamma_1$  such that for arbitrary  $\gamma > \gamma_1$ , the terms of

$X_1$  and  $X_5$  can be absorbed by  $64\delta_0^2 s^3 \gamma^4 \int_{\Omega} \varphi^3 |z|^2 dx dy dt$ , and we have

$$\begin{aligned} C_2 \int_{\Omega} |P_s z + is\partial_t \varphi z|^2 dx dy dt & \geq \int_{\Omega} |P_s^+ z|^2 dx dy dt + \int_{\Omega} |P_s^- z|^2 dx dy dt \\ & \quad + 2s\gamma C_3 \int_{\Omega} \varphi |\nabla_x z|^2 dx dy dt + 2s\gamma C_4 \int_{\Omega} \varphi |\nabla_y z|^2 dx dy dt \\ & \quad + 64\delta_0^2 s^3 \gamma^4 \int_{\Omega} \varphi^3 |z|^2 dx dy dt + B_0 + X_2 + X_6. \end{aligned}$$

Since  $\varphi > 0$  on  $\bar{\Omega}$  for  $\gamma > \gamma_1$ , there exist constants  $C_5 = C_5(\gamma)$  and  $s_1 = s_1(\gamma)$

such that for all  $s > s_1$ ,

$$\begin{aligned} C_2 \int_{\Omega} |P_s z + is\partial_t \varphi z|^2 dx dy dt & \geq \int_{\Omega} |P_s^+ z|^2 dx dy dt + \int_{\Omega} |P_s^- z|^2 dx dy dt \\ & \quad + C_5(\gamma) s \int_{\Omega} |\nabla_x z|^2 dx dy dt + C_5(\gamma) s \int_{\Omega} |\nabla_y z|^2 dx dy dt \\ & \quad + C_5(\gamma) s^3 \int_{\Omega} |z|^2 dx dy dt + B_0 + X_2 + X_6. \end{aligned}$$

Then we choose  $s_2 = s_2(\gamma) > 0$  such that  $\forall s > s_2$  all the terms of  $X_2$  and  $X_6$  can be absorbed into  $\|P_s^+ z\|_{L^2(\Omega)}^2, \|P_s^- z\|_{L^2(\Omega)}^2, C_5 \|\nabla_x z\|_{L^2(\Omega)}^2, C_5 \|\nabla_y z\|_{L^2(\Omega)}^2$  and  $C_5 s^3 \|z\|_{L^2(\Omega)}^2$ . Therefore  $\int_{\Omega} |P_s z + is \partial_t \varphi z|^2 dx dy dt \leq 2 \int_{\Omega} |P_s z|^2 dx dy dt + C_6 s^2 \int_{\Omega} |z|^2 dx dy dt$ , taking  $s > 0$  sufficiently large, we have

$$\begin{aligned} C_7 \int_{\Omega} |P_s z|^2 dx dy dt &\geq \int_{\Omega} |P_s^+ z|^2 dx dy dt + \int_{\Omega} |P_s^- z|^2 dx dy dt \\ &\quad + s \int_{\Omega} |\nabla_x z|^2 dx dy dt + s \int_{\Omega} |\nabla_y z|^2 dx dy dt \\ &\quad + s^3 \int_{\Omega} |z|^2 dx dy dt + B_0. \end{aligned}$$

Since  $z = 0$  on  $\Gamma_x \cup \Gamma_y$ ,  $\nabla_y z = 0$  and  $\nabla_x z = (\partial_\nu z) \cdot \nu$  on  $\Gamma_x$ , all the integrations on  $\Gamma_y$  vanish in (3.28), we have

$$\begin{aligned} B_0 &= -4 \operatorname{Re} \int_{\Gamma_x} s \gamma \varphi (\partial_\nu z) \nabla_x \psi \cdot \nabla_x \bar{z} dS_x dy dt + \int_{\Gamma_x} 2s \gamma \varphi (\partial_\nu \psi) |\nabla_x z|^2 dS_x dy dt \\ &= -8 \int_{\Gamma_x} s \gamma \varphi |\partial_\nu z|^2 (x - x_0) \cdot \nu dS_x dy dt + 4 \int_{\Gamma_x} s \gamma \varphi |\partial_\nu z|^2 (x - x_0) \cdot \nu dS_x dy dt \\ &= -4 \int_{\Gamma_x} s \gamma \varphi |\partial_\nu z|^2 (x - x_0) \cdot \nu dS_x dy dt \\ &\geq -4 \int_{\Gamma_x \cap \{(x - x_0) \cdot \nu \geq 0\}} s \gamma \varphi |\partial_\nu z|^2 (x - x_0) \cdot \nu dS_x dy dt. \end{aligned}$$

We obtain

$$\begin{aligned} &\int_{\Omega} |P_s^+ z|^2 dx dy dt + \int_{\Omega} |P_s^- z|^2 dx dy dt + s \int_{\Omega} |\nabla_x z|^2 dx dy dt \\ &\quad + s \int_{\Omega} |\nabla_y z|^2 dx dy dt + s^3 \int_{\Omega} |z|^2 dx dy dt. \\ &\leq C_8 \int_{\Omega} |P_s z|^2 dx dy dt + C_8 s \int_{\partial D_+ \times G \times (-T, T)} |\partial_\nu z|^2 dS_x dy dt. \end{aligned}$$

Finally, we rewrite our inequality with  $z$  instead of  $u$ . By the relation

$$\begin{aligned}
|z|^2 &= e^{2s\varphi} |u|^2, \quad |\partial_\nu z|^2 = |\partial_\nu u|^2 e^{2s\varphi} \text{ on } \partial D_+ \times G \times (-T, T), \\
|\nabla_x u e^{s\varphi}|^2 &= |\nabla_x z - s\lambda\varphi e^{s\varphi} u \nabla_x \psi|^2 \leq 2|\nabla_x z|^2 + 2s^2 \lambda^2 \varphi^2 |\nabla_x \psi|^2 |z|^2, \\
|\nabla_y u e^{s\varphi}|^2 &= |\nabla_y z - s\lambda\varphi e^{s\varphi} u \nabla_y \psi|^2 \leq 2|\nabla_y z|^2 + 2s^2 \lambda^2 \varphi^2 |\nabla_y \psi|^2 |z|^2, \\
|L_0 u|^2 &\leq 2|Lu|^2 + 2 \left| \sum_{i=1}^n a_i(x, y, t) u_{x_i} + \sum_{j=1}^m b_j(x, y, t) u_{y_j} + a_0(x, y, t) u(x, y, t) \right|^2,
\end{aligned}$$

we see that there exist positive constants  $C_9 = C_9(\gamma)$  and  $s_0 > s_2(\gamma)$  such that for all  $s > s_0$ ,

$$\begin{aligned}
&\int_{\Omega} (s|\nabla_y u|^2 + s|\nabla_x u|^2 + s^3|u|^2) e^{2s\varphi} dx dy dt \\
&\leq C_9 \int_{\Omega} |Lu|^2 e^{2s\varphi} dx dy dt + C_9 \int_{\partial D_+ \times G \times (-T, T)} s |\partial_\nu u|^2 e^{2s\varphi} dS_x dy dt.
\end{aligned}$$

We complete the proof of Theorem 1.

#### 4. PROOF OF THEOREM 1

Since  $u$  itself does not satisfy (3.6), in order to apply Proposition 1, we have to introduce a cut-off function. Moreover, we need to introduce several notations. We set

$$\tilde{r} = \max_{x \in \overline{D}} |x - x_0|, \quad (4.1)$$

$$r = \min_{x \in \overline{D}} |x - x_0|. \quad (4.2)$$

By  $x_0 \notin \overline{D}$ , we see that  $r > 0$ . We choose  $\rho > 1$  sufficiently large so that

$$\frac{\tilde{r}}{r} < \rho. \quad (4.3)$$

By (4.3) and the assumption on  $L$ , we have

$$\frac{\alpha L^2}{\rho^2} < r^2 < \tilde{r}^2 < \alpha L^2. \quad (4.4)$$

Furthermore, if necessary, we choose smaller  $\alpha$  such that

$$r^2 > \alpha^2 L^2. \quad (4.5)$$

We arbitrarily choose  $y_0 = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m$  satisfying

$$|y_0| \leq L - \frac{L}{\rho} - \epsilon. \quad (4.6)$$

We set

$$\begin{aligned} G_1 &= \{y \in \mathbb{R}^m; |y - y_0| < L\}, \\ G_2 &= \{y \in \mathbb{R}^m; |y| < 2L\}, \\ \Omega_0 &= D \times G_1 \times \{t = 0\}, \quad \Omega_1 = D \times G_1 \times (-T, T), \\ \Omega_2 &= D \times G_2 \times (-T, T). \end{aligned}$$

Then (4.1), (4.2) and (4.4) yields, if  $x \in D$  and  $|y - y_0| \leq L$ ,

$$\begin{aligned} \psi(x, y, \mp T) &= |x - x_0|^2 - \alpha |y - y_0|^2 - \beta T^2 \\ &\leq |x - x_0|^2 - \alpha |y - y_0|^2 \\ &\leq \tilde{r}^2 - \alpha L^2 \\ &< 0 \end{aligned} \quad (4.7)$$

and if  $x \in D$ ,  $|y - y_0| \leq L$  and  $|t| < T$ ,

$$\begin{aligned} \psi(x, y, t) &= |x - x_0|^2 - \alpha |y - y_0|^2 - \beta |t|^2 \\ &\leq |x - x_0|^2 - \alpha |y - y_0|^2 \\ &\leq \tilde{r}^2 - \alpha L^2 \\ &< 0 \end{aligned} \quad (4.8)$$



and if  $x \in D$  and  $|y - y_0| \leq \frac{L}{\rho}$ ,

$$\begin{aligned}\psi(x, y, 0) &= |x - x_0|^2 - \alpha |y - y_0|^2 \\ &\geq r^2 - \alpha \frac{L^2}{\rho^2} \\ &> 0.\end{aligned}\tag{4.9}$$

Therefore, for small  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$\psi(x, y, t) < -\epsilon, \quad x \in D, \tag{4.10}$$

if  $T - 2\delta \leq |t| \leq T$  or  $L - 2\delta \leq |y - y_0| \leq L$  and

$$\psi(x, y, t) > \epsilon, \quad x \in D, \quad |t| < \delta, \quad |y - y_0| \leq \frac{L}{\rho}. \tag{4.11}$$

Let us define a cut-off function  $\chi(y, t) = \chi_0(t) \chi_0(|y - y_0|)$ , where  $\chi_0 \in C_0^\infty(\mathbb{R})$  such that  $0 \leq \chi_0 \leq 1$  and

$$\begin{aligned}\chi_0(t) &= \begin{cases} 0, & T - \delta \leq |t| \leq T, \\ 1, & |t| \leq T - 2\delta, \end{cases} \\ \chi_0(|y - y_0|) &= \begin{cases} 0, & L - \delta \leq |y - y_0| \leq L, \\ 1, & |y - y_0| \leq L - 2\delta. \end{cases}\end{aligned}$$

Then we see that  $\chi \in C_0^\infty(\mathbb{R}^{m+1})$ ,  $0 \leq \chi \leq 1$  and

$$\chi(y, t) = \begin{cases} 0, & T - \delta \leq |t| \leq T \text{ or } L - \delta \leq |y - y_0| \leq L, \\ 1, & |t| \leq T - 2\delta \text{ and } |y - y_0| \leq L - 2\delta. \end{cases} \tag{4.12}$$

By choosing  $\delta > 0$  smaller if necessary, we assume

$$\frac{L}{\rho} < L - 2\delta. \tag{4.13}$$

We set

$$w_k = (\partial_t^k u) \chi, \quad k = 1, 2.$$

Then

$$\begin{aligned} Aw_k &= f \partial_t^k R \chi + 2 (\nabla_y \partial_t^k u \cdot \nabla_y \chi) + \partial_t^k u (\Delta_y \chi + i \partial_t \chi), \\ x &\in D, \quad y \in G_1, \quad k = 1, 2 \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} w_k(x, y, t) &= |\nabla_y w_k(x, y, t)| = 0, \quad (x, y, t) \in D \times \partial G_1 \times (-T, T), \\ w_k(x, y, t) &= 0, \quad (x, y, t) \in \partial D \times G_1 \times (-T, T), \\ w_k(x, y, T) &= w_k(x, y, -T) = 0, \quad (x, y, t) \in D \times G_1. \end{aligned} \quad (4.15)$$

From (4.6) we note that

$$G_1 \subset G_2. \quad (4.16)$$

By (4.14)-(4.15), we can apply the Carleman estimate (see Proposition 1) to  $w_1, w_2$  :

$$\begin{aligned} & \int_{\Omega_1} \sum_{k=1}^2 \left( s |\nabla_x w_k|^2 + s |\nabla_y w_k|^2 + s |w_k|^2 \right) e^{2s\varphi} dx dy dt \\ & \leq C \int_{\Omega_1} \sum_{k=1}^2 \chi^2 f^2 |\partial_t^k R|^2 e^{2s\varphi} dx dy dt \\ & \quad + C \int_{\Omega_1} \sum_{k=1}^2 \left( |2 (\nabla_y \partial_t^k u \cdot \nabla_y \chi) + \partial_t^k u (\Delta_y \chi + i \partial_t \chi)|^2 \right) e^{2s\varphi} dx dy dt \\ & \quad + C \int_{\partial D_+ \times G_1 \times (-T, T)} \sum_{k=1}^2 s |\partial_\nu w_k|^2 e^{2s\varphi} dS_x dy dt \\ & = S_1 + S_2 + S_3. \end{aligned} \quad (4.17)$$

Here and henceforth,  $C > 0$  denotes a generic constant which is independent of

$s > 0$ . From the assumption on  $R$ , we have

$$\begin{aligned} S_1 &= C \int_{\Omega_1} \sum_{k=1}^2 \chi^2 f^2 |\partial_t^k R|^2 e^{2s\varphi} dx dy dt \\ &\leq C \int_{\Omega_1} \chi^2 f^2 e^{2s\varphi} dx dy dt. \end{aligned}$$

By (4.12), we see that  $\partial_t \chi = 0$  for  $|t| \leq T - 2\delta$  or  $T - \delta \leq |t| \leq T$  and  $|\nabla_y \chi| = \Delta_y \chi = 0$  for  $|y - y_0| \leq L - 2\delta$  or  $L - \delta \leq |y - y_0| \leq L$ . Therefore, if  $|t| \in [0, T - 2\delta] \cup [T - \delta, T]$  and  $|y - y_0| \in [0, L - 2\delta] \cup [L - \delta, L]$ , then  $\partial_t \chi = |\nabla_y \chi| = \Delta_y \chi = 0$ . Hence

$$\begin{aligned} S_2 &= C \left( \int_{\{T-2\delta \leq |t| \leq T-\delta\} \cap \Omega_1} \sum_{k=1}^2 |2 (\nabla_y \partial_t^k u \cdot \nabla_y \chi) \right. \\ &\quad \left. + \partial_t^k u (\Delta_y \chi + i \partial_t \chi) \right|^2 e^{2s\varphi} dx dy dt \Big) \\ &\quad + C \left( \int_{\{L-2\delta \leq |y-y_0| \leq L-\delta\} \cap \Omega_1} \sum_{k=1}^2 |2 (\nabla_y \partial_t^k u \cdot \nabla_y \chi) \right. \\ &\quad \left. + \partial_t^k u (\Delta_y \chi + i \partial_t \chi) \right|^2 e^{2s\varphi} dx dy dt \Big) \\ &= C \left( \int_{\{T-2\delta \leq |t| \leq T-\delta\} \cap \Omega_1} \sum_{k=1}^2 |2 (\nabla_y \partial_t^k u \cdot \nabla_y \chi) \right. \\ &\quad \left. + \partial_t^k u (\Delta_y \chi + i \partial_t \chi) \right|^2 (\exp(2se^{-\gamma\epsilon})) dx dy dt \Big) \\ &\quad + C \left( \int_{\{L-2\delta \leq |y-y_0| \leq L-\delta\} \cap \Omega_1} \sum_{k=1}^2 |2 (\nabla_y \partial_t^k u \cdot \nabla_y \chi) \right. \\ &\quad \left. + \partial_t^k u (\Delta_y \chi + i \partial_t \chi) \right|^2 (\exp(2se^{-\gamma\epsilon})) dx dy dt \Big) \\ &\leq C \int_{\Omega_1} \sum_{k=1}^2 \left( |\partial_t^k u|^2 + |\nabla_y \partial_t^k u|^2 \right) e^{2s\kappa_1} dx dy dt \\ &\leq CM^2 e^{2s\kappa_1}. \end{aligned} \tag{4.18}$$

Here and henceforth we set

$$\kappa_1 = e^{-\gamma\epsilon}, \quad \kappa_2 = e^{\gamma\epsilon}.$$

Finally, we have

$$\begin{aligned}
S_3 &= C \int_{\partial D_+ \times G_1 \times (-T, T)} \sum_{k=1}^2 s |\partial_\nu w_k|^2 e^{2s\varphi} dS_x dy dt \\
&\leq C e^{cs} \int_{\partial D_+ \times G_2 \times (-T, T)} \sum_{k=1}^2 |\partial_\nu \partial_t^k u|^2 dS_x dy dt := C e^{cs} d^2, \quad (4.19)
\end{aligned}$$

where

$$d^2 = \int_{\partial D_+ \times G_2 \times (-T, T)} \sum_{k=1}^2 |\partial_\nu \partial_t^k u|^2 dS_x dy dt.$$

As a result, (4.17) yields

$$\begin{aligned}
&\int_{\Omega_1} \sum_{k=1}^2 \left( s |\nabla_x w_k|^2 + s |\nabla_y w_k|^2 + s^3 |w_k|^2 \right) e^{2s\varphi} dx dy dt \\
&\leq C \int_{\Omega_1} \chi^2 f^2 e^{2s\varphi} dx dy dt + CM^2 e^{2s\kappa_1} + C e^{cs} d^2. \quad (4.20)
\end{aligned}$$

Now, by using the fact that  $\chi(y, -T) = 0$  for  $y \in G_1$  by (4.12), we can write

$$\begin{aligned}
&\int_{\Omega_0} |\chi(y, 0)|^2 |i \partial_t u(x, y, 0)|^2 e^{2s\varphi(x, y, 0)} dx dy \\
&= \int_{-T}^0 \partial_t \left( \int_{\Omega_0} \chi^2 |\partial_t u(x, y, t)|^2 e^{2s\varphi(x, y, t)} dx dy \right) dt \\
&= \int_{-T}^0 \int_{\Omega_0} \left( 2\chi \partial_t \chi |\partial_t u|^2 + \chi^2 \partial_t (|\partial_t u|^2) + \chi^2 |\partial_t u|^2 2s \partial_t \varphi \right) e^{2s\varphi(x, y, t)} dx dy dt \\
&= \int_{-T}^0 \int_{\Omega_0} \left( 2\chi \partial_t \chi |\partial_t u|^2 + 2\chi^2 \operatorname{Re} \partial_t^2 u \partial_t \bar{u} + \chi^2 |\partial_t u|^2 2s \partial_t \varphi \right) e^{2s\varphi(x, y, t)} dx dy dt \\
&\leq \int_{-T}^T \int_{\Omega_0} |\chi \partial_t u|^2 e^{2s\varphi} dx dy dt + \int_{-T}^T \int_{D \times G_1} |\partial_t \chi \partial_t u|^2 e^{2s\varphi} dx dy dt \\
&\quad + \int_{-T}^T \int_{D \times G_1} |\chi \partial_t u|^2 e^{2s\varphi} dx dy dt + \int_{-T}^T \int_{\Omega_0} |\chi \partial_t^2 u|^2 e^{2s\varphi} dx dy dt \\
&\quad + s \int_{-T}^T \int_{\Omega_0} |\chi \partial_t u|^2 e^{2s\varphi} dx dy dt
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{-T}^T \int_{\Omega_0} |\partial_t \chi \partial_t u|^2 e^{2s\varphi} dx dy dt \\
&\quad + C \int_{-T}^T \int_{\Omega_0} \left( |\chi \partial_t u|^2 + |\chi \partial_t^2 u|^2 + s |\chi \partial_t u|^2 \right) e^{2s\varphi} dx dy dt \\
&\leq CM^2 e^{2s\kappa_1} + C \int_{\Omega_1} \left( |w_1|^2 + |w_2|^2 + s |w_1|^2 \right) e^{2s\varphi} dx dy dt.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\Omega_0} |\chi(y, 0)|^2 |i \partial_t u(x, y, 0)|^2 e^{2s\varphi(x, y, 0)} dx dy \\
&\leq CM^2 e^{2s\kappa_1} + C \int_{\Omega_1} \left( s |w_1|^2 + |w_2|^2 \right) e^{2s\varphi} dx dy dt.
\end{aligned}$$

Then, applying (4.20), we obtain

$$\begin{aligned}
&\int_{\Omega_0} |\chi_0(|y - y_0|)|^2 |i \partial_t u(x, y, 0)|^2 e^{2s\varphi(x, y, 0)} dx dy \\
&\leq \frac{C}{s} \int_{\Omega_1} \chi^2(y, t) |f|^2 e^{2s\varphi(x, y, t)} dx dy dt + CM^2 e^{2s\kappa_1} + C e^{cs} d^2 \\
&= \frac{C}{s} \int_{\Omega_1} \chi_0^2(t) \chi_0^2(|y - y_0|) |f|^2 e^{2s\varphi(x, y, t)} dx dy dt + CM^2 e^{2s\kappa_1} + C e^{cs} d^2 \\
&\leq \frac{C}{s} \int_{\Omega_0} \chi_0^2(|y - y_0|) |f|^2 e^{2s\varphi(x, y, 0)} dx dy + CM^2 e^{2s\kappa_1} + C e^{cs} d^2, \quad (4.21)
\end{aligned}$$

where we used  $|\chi_0(t)| \leq 1$  and  $e^{2s\varphi(x, y, t)} \leq e^{2s\varphi(x, y, 0)}$  for  $x \in D$  and  $y \in G_1$ .

On the other hand, by substituting  $t = 0$  in (2.1) and applying  $u(x, y, 0) = 0$  and  $R(x, y, 0) \neq 0$ , for  $x \in \overline{D}$  and  $|y| \leq 2L$ , we get

$$f(x, y) = \frac{i \partial_t u(x, y, 0)}{R(x, y, 0)}, \quad x \in \overline{D}, \quad |y| \leq 2L. \quad (4.22)$$

By applying (4.22) in (4.21), we have

$$\begin{aligned}
&\int_{\Omega_0} \chi_0^2(|y - y_0|) |f|^2 e^{2s\varphi(x, y, 0)} dx dy \\
&\leq \frac{C}{s} \int_{\Omega_0} \chi_0^2(|y - y_0|) |f|^2 e^{2s\varphi(x, y, 0)} dx dy + CM^2 e^{2s\kappa_1} + C e^{cs} d^2
\end{aligned}$$

for all large  $s > 0$ . Now, we absorb the first term on the right-hand side into the left-hand side by choosing  $s > 0$  large, we get

$$\int_{\Omega_0} \chi_0^2(|y - y_0|) |f|^2 e^{2s\varphi(x,y,0)} dx dy \leq CM^2 e^{2s\kappa_1} + Ce^{cs} d^2$$

for all large  $s > 0$ .

Replacing the integration domain on the left-hand side by  $D \times \left\{ y; |y - y_0| < \frac{L}{\rho} \right\} \subset \Omega_0$  and using the facts that  $\chi_0(|y - y_0|) = 1$  in  $D \times \left\{ y; |y - y_0| < \frac{L}{\rho} \right\}$  and

$$e^{2s\varphi(x,y,0)} = \exp\left(2se^{\gamma\psi(x,y,0)}\right) > \exp(2se^{\gamma\epsilon}) = e^{2s\kappa_2},$$

we obtain,

$$e^{2s\kappa_2} \int_{D \times \left\{ y; |y - y_0| < \frac{L}{\rho} \right\}} |f|^2 dx dy \leq CM^2 e^{2s\kappa_1} + Ce^{cs} d^2$$

for all  $s \geq s_0$ , where  $s_0$  is some constant. Since  $\kappa_2 > \kappa_1$ , the last inequality implies

$$\int_{D \times \left\{ y; |y - y_0| < \frac{L}{\rho} \right\}} |f|^2 dx dy \leq CM^2 e^{-2s\kappa} + Ce^{cs} d^2 \quad (4.23)$$

for all  $s \geq s_0$ , where  $\kappa = \kappa_2 - \kappa_1 > 0$ . We separately consider the two cases:

Case 1. Let  $M \geq d$ . Choosing  $s \geq 0$  such that

$$M^2 e^{-2s\kappa} = e^{Cs} d^2, \text{ that is, } s = \frac{2}{C + 2\kappa} \log \frac{M}{d} \geq 0,$$

we obtain

$$\int_{D \times \left\{ y; |y - y_0| < \frac{L}{\rho} \right\}} |f|^2 dx dy \leq 2M^{\frac{2C}{C+2\kappa}} d^{\frac{4\kappa}{C+2\kappa}}.$$

Case 2. Let  $M < d$ . Then setting  $s = 0$  in (4.23) we have

$$\int_{D \times \left\{ y; |y - y_0| < \frac{L}{\rho} \right\}} |f|^2 dx dy \leq 2Cd^2.$$

Therefore we can choose  $\theta \in (0, 1)$  such that

$$\int_{D \times \{y; |y-y_0| < \frac{L}{\rho}\}} |f|^2 dx dy \leq C (d^{2\theta} + d^2)$$

for all  $y_0 \in \mathbb{R}^m$  satisfying  $|y_0| < L - \frac{L}{\rho} - \epsilon$ . By  $\|\partial_t u\|_{H^2(D \times \{|y| < 2L\} \times (-T, T))} \leq M$ , the trace theorem yields  $d \leq CM$ , which implies  $d \leq Cd^\theta$ . Varying  $y_0$  and noting

$$\bigcup \left\{ y \in \mathbb{R}^m; |y - y_0| \leq \frac{L}{\rho}, |y_0| < L - \frac{L}{\rho} - \epsilon \right\} = \{y \in \mathbb{R}^m; |y| < L - \epsilon\},$$

we obtain

$$\int_{D \times \{y; |y| < L - \epsilon\}} |f(x, y)|^2 dx dy \leq Cd^{2\theta}.$$

Thus the proof of Theorem 1 is completed.

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